Matemáticas VII

Preparaduría

Semana #04

* ECUACIONES DIFERENCIALES

NOTACIÓN

$$L = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0 I \; ; \quad \text{donde} \quad D^k = \frac{\partial^k}{\partial x^k} \; ; \; a_k \in C^\infty \wedge a_n \neq 0.$$

Se denomina operador Diferencial en sentido generalizado.

Se busca resolver ecuaciones de la siguiente forma

$L_{gen}\, U(x) = F(x)$; $F(x) \in \mathcal{D}'(V)$; F es dada y $U(x)$ es la incognita

NOTA:
Si $a_n ; a_{n-1} ; \dots a_1 ; a_0$ son números la ecuación diferencial se denomina Ecuación Diferencial generalizada a coeficientes constantes. Sumado a ello si $F(x) = 0$ se le agrega el término Homogenea a su nombre. Es decir:
Ecuación Diferencial generalizada a coeficientes constantes homogenea.

OBSERVACIÓN:
Si L es de coeficientes constantes la unica solución de $L_{gen}(U(x)) \equiv$ son funciones clasicas $U \in C^\infty$.

Solución Fundamental (S.F). ó Función de Green.

Definimos la solución fundamental ó función de Green de L; como la distribución G tal que:

$$\boxed{L_{gen}(G(x)) = \delta(x).}$$

La solución fundamental general de L se define por $E(x) = G(x) + U_h$; donde U_h es la solución general de la ecuación homogenea asociada $L_{gen}(U(x)) \equiv 0$.

OBSERVACIÓN:
Si L es de coeficientes constantes, EXISTE $g \in C^\infty$ ÚNICA tal que :

$$\boxed{G(x) = g(x)\, H(x)} \quad y \quad \boxed{L_{gen}(G(x)) = \delta(x)}$$

$\boxed{G(x) = g(x)\, H(x)}$ Se denomina propagador CAUSAL

NOTA:
Al buscar el propagador causal el Heaviside debe estar centrada en el punto donde se dan las condiciones de valores iniciales. Si por el contrario no se nos da condiciones iniciales o comportamiento de la función buscada este Heaviside estaria centrado en cero

EJERCICIOS:

1) Resolver el siguiente problema de valores iniciales.

$$\begin{cases} y^{IV}(x) - y(x) = -x \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 0 \\ y'''(0) = 1 \end{cases}$$

Solución

→ Paso #01: Consideramos la función causal. $\boxed{(u(x) = y(x)H(x-a))}$

$$\Rightarrow \boxed{u(x) = y(x)H(x)}$$

→ Paso #02: Consideramos la ecuación diferencial (distribucional)

$$\boxed{L_{gen}(u(x)) = F(x)}$$

Donde $\boxed{L = D^4 - I}$ $\Rightarrow$ $\boxed{L_{gen}(u(x)) = u^{IV}_{gen}(x) - u(x)}$

→ Paso #03: Sustituimos las derivadas generalizadas, partiendo de:

$$\boxed{u(x) = y(x)H(x)}$$

$$\Rightarrow u'_{gen}(x) = y'(x)H(x) + y(x)(H(x))' \; ; \quad (H(x))' = \delta(x)$$

$$= y'(x)H(x) + y(x)\delta(x)$$

$$\langle y(x)\delta(x) | \varphi(x)\rangle = \langle \delta(x) | y(x)\varphi(x)\rangle = \underbrace{y(0)}_{=0}\varphi(0) = 0$$

$$\Rightarrow y(x)\delta(x) = 0$$

$$\Rightarrow \boxed{u'_{gen}(x) = y'(x)H(x)}$$

$$\Rightarrow u''_{gen}(x) = y''(x)H(x) + y'(x)(H(x))'$$

$$= y''(x)H(x) + y'(x)\delta(x) \; ;$$

$$\langle y'(x)\delta(x) | \varphi(x)\rangle = \langle \delta(x) | y'(x)\varphi(x)\rangle = \overbrace{y'(0)}^{=0}\varphi(0) = 0$$

$$\Rightarrow y'(x)\delta(x) = 0$$

$$\Rightarrow \boxed{u''_{gen}(x) = y''(x)H(x)}$$

$$\Rightarrow u'''_{gen}(x) = y'''(x)H(x) + y''(x)(H(x))'$$

$$= y'''(x)H(x) + y''(x)\delta(x) \; ;$$

$$\langle y''(x)\delta(x) | \varphi(x)\rangle = \langle \delta(x) | y''(x)\varphi(x)\rangle = \overbrace{y''(0)}^{=0}\varphi(0) = 0$$

$$\Rightarrow y''(x)\delta(x) = 0$$

$$\Rightarrow \boxed{u'''_{gen}(x) = y'''(x)H(x)}$$

$$\Rightarrow u^{IV}_{gen}(x) = y^{IV}(x)H(x) + y'''(x)(H(x))'$$

$$= y^{IV}(x)H(x) + y'''(x)\delta(x) \; ;$$

$$\langle y'''(x)\delta(x) | \varphi(x)\rangle = \langle \delta(x) | y'''(x)\varphi(x)\rangle = \overbrace{y'''(0)}^{=1}\varphi(0) = \varphi(0)$$

$$\Rightarrow y'''(x)\delta(x) = \delta(x)$$

$$\Rightarrow \boxed{u^{IV}_{gen}(x) = y^{IV}(x)H(x) + \delta(x)}$$

Sustituyendo.

$$L_{gen}(u(x)) = u^{IV}_{gen}(x) - u(x) = y^{IV}_{(x)} H(x) + \delta(x) - y(x) H(x) = \underbrace{(y^{IV}_{(x)} - y(x))}_{-x} H(x) + \delta(x).$$

$$\Rightarrow \boxed{L_{gen}(u(x)) = -x H(x) + \delta(x)} \dots (1)$$

→ **PASO #04:** Resolvemos (Escritura formal).

$L_{gen}(u(x)) = F(x)$ con $F(x) = -x H(x) + \delta(x)$.

$\boxed{u(x) = (G * F)(x)}$; $\boxed{L_{gen}(G(x)) = \delta(x).}$

→ **PASO #05:** Resolvemos (Aplicando Transformada de Laplace).

$$(1) \Rightarrow L_{gen}(u(x)) = -x H(x) + \delta(x) \Longrightarrow u^{IV}_{gen}(x) - u(x) = -x H(x) + \delta(x)$$

$$u^{IV}_{gen}(x) - u(x) = -x H(x) + \delta(x) \xrightarrow{\mathcal{L}} z^4 U(z) - U(z) = -\frac{1}{z^2} + 1$$

$$U(z)(z^4 - 1) = -\frac{1}{z^2} + 1 \Rightarrow U(z)(z^4 - 1) = \frac{z^2 - 1}{z^2} \Rightarrow \boxed{U(z) = \frac{z^2 - 1}{z^2(z^4 - 1)}}$$

$$z^4 - 1 = (z^2 + 1)(z^2 - 1) \Rightarrow U(z) = \frac{\cancel{z^2 - 1}}{z^2(z^2+1)\cancel{(z^2 - 1)}} = \frac{1}{z^2(z^2+1)}$$

$$\frac{1}{z^2(z^2+1)} = \frac{A}{z^2} + \frac{B}{z^2+1} \;;\; A(z^2+1) + Bz^2 = 1 \approx (A+B)z^2 + A = 1 \Rightarrow \boxed{A = -B}$$

$$A = 1 \;\wedge\; B = -1$$

$$\Rightarrow \boxed{U(z) = \frac{1}{z^2} - \frac{1}{z^2+1}}$$

→ **PASO #06:** Aplicamos antitransformada y obtenemos $u(x)$.

$$U(z) \xrightarrow{\mathcal{L}^{-1}} u(x) \Rightarrow \frac{1}{z^2} - \frac{1}{z^2+1} \xrightarrow{\mathcal{L}^{-1}} x H(x) - H(x) \operatorname{SEN}(x)$$

$$\Rightarrow \boxed{u(x) = x H(x) - H \operatorname{SEN}(x) = (x - \operatorname{SEN}(x)) H(x).}$$

Pero $u(x) = y(x) H(x)$

$$\Rightarrow \boxed{y(x) = x - \operatorname{SEN}(x).}$$

2) $\begin{cases} y''(x) - 2y'(x) - 3y(x) = 2 \\ y(1) = 1 \\ y'(1) = 2 \end{cases}$

$L_{gen}(u(x)) = F(x)$

$L_{gen}(u(x)) = D^2 - 2D - 3I$

$L_{gen}(u(x)) = u''_{gen}(x) - 2u'_{gen}(x) - 3u(x)$

Solución

$u(x) = y(x)H(x-1)$

$\rightarrow u'_{gen}(x) = y'(x)H(x-1) + y(x)(H(x-1))'$; $(H(x-1))' = \delta(x-1)$

$= y'(x)H(x-1) + y(x)\delta(x-1)$

$\langle y(x)\delta(x-1)|\varphi(x)\rangle = \langle \delta(x-1)|y(x)\varphi(x)\rangle = \underbrace{y(1)}_{=1}\varphi(1) = \varphi(1) = \langle \delta(x-1)|\varphi(x)\rangle \Rightarrow y(x)\delta(x-1) = \delta(x-1)$

$\Rightarrow \boxed{u'_{gen}(x) = y'(x)H(x-1) + \delta(x-1)}$

$\rightarrow u''_{gen}(x) = y''(x)H(x-1) + y'(x)(H(x-1))' + \delta'(x-1)$

$= y''(x)H(x-1) + y'(x)\delta(x-1) + \delta'(x-1)$

$\langle y'(x)\delta(x-1)|\varphi(x)\rangle = \langle \delta(x-1)|y'(x)\varphi(x)\rangle = \underbrace{y'(1)}_{=2}\varphi(1) = 2\varphi(1) = 2\langle \delta(x-1)|\varphi(x)\rangle \Rightarrow y'(x)\delta(x-1) = 2\delta(x-1)$

$\Rightarrow \boxed{u''_{gen}(x) = y''(x)H(x-1) + 2\delta(x-1) + \delta'(x-1)}$

$\Rightarrow u''_{gen}(x) - 2u'_{gen}(x) - 3u(x) = y''(x)H(x-1) + \cancel{2\delta(x-1)} + \delta'(x-1) - 2y'(x)H(x-1) - \cancel{2\delta(x-1)} - 3y(x)H(x-1)$

$= \underbrace{(y''(x) - 2y'(x) - 3y(x))}_{2}H(x-1) + \delta'(x-1)$

$\Rightarrow \boxed{L_{gen}(u(x)) = 2H(x-1) + \delta'(x-1)}$

$\Rightarrow u''_{gen}(x) - 2u'_{gen}(x) - 3u(x) = 2H(x-1) + \delta'(x-1)$

$\xrightarrow{\mathcal{L}} z^2U(z) - 2zU(z) - 3U(z) = \frac{2}{z}e^{-z} + ze^{-z} \Rightarrow U(z)(z^2 - 2z - 3) = e^{-z}\left(\frac{2}{z} + z\right) = e^{-z}\left(\frac{2+z^2}{z}\right)$

$\Rightarrow \boxed{U(z) = \left[\frac{2+z^2}{z(z^2-2z-3)}\right]e^{-z}}$

$$U(z) = V(z)e^{-z}$$

$$\Rightarrow V(z) = \frac{2+z^2}{z(z^2-2z-3)}$$

$$z^2 - 2z - 3$$

$$\begin{array}{r|rrr}
 & 1 & -2 & -3 \\
3 & & 3 & 3 \\
\hline
 & 1 & 1 & \boxed{0} \\
-1 & & -1 & \\
\hline
 & 1 & \boxed{0} &
\end{array}$$

$$z^2 - 2z - 3 = (z-3)(z+1)$$

$$\Rightarrow \boxed{V(z) = \frac{2+z^2}{z(z-3)(z+1)}}$$

$$\Rightarrow \frac{z^2+2}{z(z-3)(z+1)} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z+1}$$

$$\Rightarrow z^2+2 = A(z-3)(z+1) + B(z)(z+1) + C(z)(z-3)$$

- $z=0 \Rightarrow 2 = A(-3)(1) \Rightarrow \boxed{A = -2/3}$
- $z=3 \Rightarrow 11 = B(3)(4) \Rightarrow \boxed{B = 11/12}$
- $z=-1 \Rightarrow 3 = C(-1)(-4) \Rightarrow \boxed{C = 3/4}$

$$\Rightarrow \frac{z^2+2}{z(z-3)(z+1)} = -\frac{2}{3}\cdot\left(\frac{1}{z}\right) + \frac{11}{12}\cdot\left(\frac{1}{z-3}\right) + \frac{3}{4}\cdot\left(\frac{1}{z+1}\right)$$

$$V(z) \xrightarrow{\mathcal{L}^{-1}} v(x). \Rightarrow \frac{z^2+2}{z(z-3)(z+1)} \xrightarrow{\mathcal{L}^{-1}} -\frac{2}{3}H(x) + \frac{11}{12}H(x)e^{3x} + \frac{3}{4}H(x)e^{-x} = v(x)$$

$$\boxed{U(z) = V(z)e^{-z} \Rightarrow V(z)e^{-z} \xrightarrow{\mathcal{L}^{-1}} v(x-1) \;; \quad u(x) = v(x-1)}$$

$$\Rightarrow u(x) = -\frac{2}{3}H(x-1) + \frac{11}{12}H(x-1)e^{3(x-1)} + \frac{3}{4}H(x-1)e^{-(x-1)}$$

$$= \left(-\frac{2}{3} + \frac{11}{12}e^{3(x-1)} + \frac{3}{4}e^{-(x-1)}\right)H(x-1) \;; \quad u(x) = y(x)H(x-1)$$

$$\Rightarrow \boxed{y(x) = -\frac{2}{3} + \frac{11}{12}e^{3(x-1)} + \frac{3}{4}e^{-(x-1)}}$$

3) $\begin{cases} x y''(x) + 2x y'(x) - 2y(x) = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

SOLUCIÓN

$L = xD^2 + 2xD - 2I$

$(H(x))' = \delta(x)$

$\Rightarrow u(x) = y(x) H(x)$

$\Rightarrow u'_{gen}(x) = y'(x) H(x) + y(x) (H(x))'$
$= y'(x) H(x) + y(x) \delta(x)$;

$\langle y(x)\delta(x) | \varphi(x) \rangle = \langle \delta(x) | y(x)\varphi(x) \rangle = \overset{=0}{\cancel{y(0)}}\varphi(0) = 0$

$\Rightarrow y(x)\delta(x) = 0$

$\Rightarrow \boxed{u'_{gen}(x) = y'(x) H(x)}$

$\Rightarrow u''_{gen}(x) = y''(x) H(x) + y'(x) (H(x))'$
$= y''(x) H(x) + y'(x) \delta(x)$;

$\langle y'(x)\delta(x) | \varphi(x) \rangle = \langle \delta(x) | y'(x)\varphi(x) \rangle = \overset{=1}{\cancel{y'(0)}}\varphi(0) = \varphi(0)$

$\Rightarrow y'(x)\delta(x) = \delta(x)$

$\Rightarrow \boxed{u''_{gen}(x) = y''(x) H(x) + \delta(x).}$

$\Rightarrow x u''_{gen}(x) + 2x u'_{gen}(x) - 2u(x) = x y''(x) H(x) + x\delta(x) + 2x y'(x) H(x) - 2y(x) H(x)$
$= \underbrace{(x y''(x) + 2x y'(x) - 2y(x))}_{=0} H(x) + x\delta(x)$

$\Rightarrow \boxed{x u''_{gen}(x) + 2x u'_{gen}(x) - 2u(x) = x\delta(x).}$

$\xrightarrow{\mathcal{L}} -2zU(z) - z^2U'(z) - 2U(z) - 2zU'(z) - 2U(z) = 0.$

$\Rightarrow \boxed{[-z^2 - 2z] U'(z) + [-2z - 4] U(z) = 0}$

$\Rightarrow \frac{U'(z)}{U(z)} = -\left(\frac{2z+4}{z^2+2z}\right) = -\frac{2}{z}\left(\cancel{\frac{z+2}{z+2}}\right)^{=1} \Rightarrow \boxed{\frac{U'(z)}{U(z)} = -\frac{2}{z}}$

$\Rightarrow \ln(U(z)) = -2\ln|z| + C$

$\Rightarrow \boxed{U(z) = \frac{A}{z^2}}$

$\frac{A}{z^2} \xrightarrow{\mathcal{L}^{-1}} AxH(x) \Rightarrow u(x) = y(x) H(x) = AxH(x) \Rightarrow \boxed{y(x) = Ax}$

Pero $\left.\begin{array}{l} y'(x) = A \\ y'(0) = 1 \end{array}\right\} \boxed{A = 1} \Rightarrow \boxed{y(x) = x}$

$\boxed{u(x) = xH(x)}$

4) Calcule el propagador causal de : $L = 9D^2 + 4I$ (Propagador causal = Función de Green).

Solución

→ Queremos calcular $G(x) = g(x)H(x)$ tal que $L_{gen}(G(x)) = \delta(x)$.

$$\Rightarrow \boxed{9\, G''_{gen}(x) + 4\, G(x) = \delta(x)}$$

→ Aplicamos transformada de Laplace.

$$9z^2 \mathcal{L}(G(x))(z) + 4\mathcal{L}(G(x))(z) = 1 \Rightarrow 9z^2 G(z) + 4\, G(z) = 1$$

$$G(z)(9z^2 + 4) = 1$$

$$\Rightarrow \boxed{G(z) = \frac{1}{9z^2+4}}$$

→ Aplicando antitransformada de Laplace.

$$G(z) = \frac{1}{9z^2+4} = \frac{2/3}{\frac{2}{3}\cdot 9 \cdot \left(z^2 + \left(\frac{2}{3}\right)^2\right)} = \frac{1}{6} \cdot \frac{2/3}{z^2 + (2/3)^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \boxed{G(x) = \left[\frac{1}{6} \operatorname{sen}\left(\frac{2}{3}x\right)\right] H(x)}$$